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The intersection cohomology of toric varieties

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Introduction

The intersection cohomology theory of Goresky and MacPherson [10], [12], which was further developed by Beilinson, Bernstein and Deligne [1], attaches to singular spaces new topological invariants much better behaved than ordinary cohomology or homology groups.

Since a toric variety is a simple-minded singular space described in terms of a fan, it is natural to try to describe its intersection cohomology group and, more generally its intersection complex, in terms of the fan as well.

There have been earlier attemps in this direction by J. N. Bernstein, A. G. Khovanskii and R. D. MacPherson (see Stanley [26]), Denef and Loeser [9] and others. However, they depend either on the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber (see Beilinson, Bernstein and Deligne [1] as well as Goresky and MacPherson [11]) or on the purity theorem of Deligne and Gabber (cf. Deligne [7], [8]).

We also resort here to the decomposition theorem to interpret, in terms of fans, the intersection cohomology groups and intersection complexes of toric varieties with respect to the middle perversity. Hopefully we will be able to find an elementary proof for these results. In this paper, we only give clues. Our problem turns out to be closely related to the problem of finding an elementary proof of the strong Lefschetz theorem for toric varieties, and hence of the elementary proof due to Stanley [25] of the "g-theorem" for simplicial convex polytopes conjectured by McMullen [18]. (See also [20], [21])

In Section 1, we quote the relevant parts of [22] to recall the definitions of the logarithmic double complexes and Ishida's complexes for toric varieties. The algebraic de

Rham theorem as well as certain vanishing theorems for them will play important roles in describing the intersection cohomology and intersection complexes for toric varieties. We also recall the Chow rings for simplicial fans which provide convenient tools later. The details will appear in Park [24].

In Section 2, we apply results in the previous section to study the intersection cohomology and intersection complex for toric varieties. After quoting relevant results valid in general, we restrict ourselves to the simplest nontrivial case of isolated non-quotient toric singularities.

1 The algebraic de Rham theorem

1.1 Logarithmic double complex

The following logarithmic double complex for a toric variety will play important roles in the description of the intersection cohomology groups and intersection complex of the toric variety.

Let Δ be a finite fan for a free $\mathbf{Z}$-module N of rank r, and denote by $X := T_N \operatorname{emb}(\Delta)$ the associated r-dimensional toric variety over the field $\mathbf{C}$ of complex numbers. The complement $D := X \setminus T_N$ of the algebraic torus $T_N := N \otimes_{\mathbf{Z}} \mathbf{C}^* \cong (\mathbf{C}^*)^r$ is a Weil divisor on X but is not necessarily a Cartier divisor. The dual $\mathbf{Z}$-module $M := \operatorname{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$, with the canonical bilinear pairing $\langle\ ,\ \rangle : M \times N \to \mathbf{Z}$, is isomorphic to the character group of the algebraic torus T_N. For each $m \in M$ we denote the corresponding character by $t^m : T_N \to \mathbf{C}^*$ (which was denoted by $\mathbf{e}(m)$ in [19]), and identify $\mathbf{C}[M] := \oplus_{m\in M}\mathbf{C}t^m$ with the group algebra of M over $\mathbf{C}$. Hence T_N is the group of $\mathbf{C}$-valued points of the group scheme $\operatorname{Spec}(\mathbf{C}[M])$.

Each $n \in N$ gives rise to a $\mathbf{C}$-derivation δ_n of $\mathbf{C}[M]$ defined by $\delta_n(t^m) := \langle m, n\rangle t^m$. Consequently, we have a canonical isomorphism to the Lie algebra

$$\mathbf{C} \otimes_{\mathbf{Z}} N \xrightarrow{\sim} \operatorname{Lie}(T_N), \qquad 1 \otimes n \mapsto \delta_n,$$

hence an $\mathcal{O}_X$-isomorphism $\mathcal{O}_X \otimes_{\mathbf{Z}} N \xrightarrow{\sim} \Theta_X(-\log D)$, where the right hand side is the sheaf of germs of algebraic vector fields on X with logarithmic zeros along the Weil divisor D. Its dual $\Omega^1_X(\log D)$ is the sheaf of germs of algebraic 1-forms with logarithmic poles along D, and we get an $\mathcal{O}_X$-isomorphism

$$\mathcal{O}_X \otimes_{\mathbf{Z}} M \xrightarrow{\sim} \Omega^1_X(\log D), \qquad 1 \otimes m \mapsto \frac{dt^m}{t^m}.$$

Taking the exterior product, we thus get an $\mathcal{O}_X$-isomorphism $\mathcal{O}_X \otimes_{\mathbf{Z}} \bigwedge^{\cdot} M \xrightarrow{\sim} \Omega^{\cdot}_X(\log D)$. The exterior differentiation d on the right hand side corresponds to the operation on the left hand side which sends a p-form $t^m \otimes m_1 \wedge \cdots \wedge m_p$ to the $(p+1)$-form $t^m \otimes m \wedge m_1 \wedge \cdots \wedge m_p$ (cf. [19, Chap. 3]).

Recall that the set of T_N-orbits in X is in one-to-one correspondence with Δ by the map which sends each $\sigma \in \Delta$ to the T_N-orbit

$$\mathrm{orb}(\sigma) = \mathrm{Spec}(\mathbf{C}[M \cap \sigma^\perp]) = T_{N/\mathbf{Z}(N\cap\sigma)}.$$

The closure $V(\sigma)$ in X of $\mathrm{orb}(\sigma)$ is known to be a toric variety with respect to a fan for the $\mathbf{Z}$-module $N/\mathbf{Z}(N \cap \sigma)$. Namely,

$$V(\sigma) = T_{N/\mathbf{Z}(N\cap\sigma)}\,\mathrm{emb}(\{(\tau + (-\sigma))/\mathbf{R}\sigma \mid \tau \in \mathrm{Star}_\sigma(\Delta)\}),$$

where $\mathbf{R}\sigma = \sigma + (-\sigma)$ is the smallest $\mathbf{R}$-subspace containing σ of $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$, while $\mathrm{Star}_\sigma(\Delta) := \{\tau \in \Delta \mid \tau \succ \sigma\}$. Hence $D(\sigma) := V(\sigma) \setminus \mathrm{orb}(\sigma)$ is a Weil divisor on $V(\sigma)$. In particular, we have $\mathrm{orb}(\{0\}) = T_N$, $D(\{0\}) = D$ and $V(\{0\}) = X$.

For each integer q with $0 \le q \le r$, denote $\Delta(q) := \{\sigma \in \Delta \mid \dim \sigma = q\}$. For each pair of integers p, q, let

$$\mathcal{L}_X^{p,q} := \bigoplus_{\sigma\in\Delta(q)} \Omega^{p-q}_{V(\sigma)}(\log D(\sigma)) = \bigoplus_{\sigma\in\Delta(q)} \mathcal{O}_{V(\sigma)} \otimes_{\mathbf{Z}} \bigwedge\nolimits^{p-q}(M \cap \sigma^\perp) \qquad \text{if} \quad 0 \le q \le p,$$

and $\mathcal{L}^{p,q} = 0$ otherwise. $d_{\mathrm{I}} : \mathcal{L}_X^{p,q} \to \mathcal{L}_X^{p+1,q}$ is defined to be the direct sum of the exterior differentiation for each $\sigma \in \Delta(q)$. We define $d_{\mathrm{II}} : \mathcal{L}_X^{p,q} \to \mathcal{L}_X^{p,q+1}$ as follows: The (σ,τ)-component of d_{II} for $\sigma \in \Delta(q)$ and $\tau \in \Delta(q+1)$ is defined to be zero when σ is not a face of τ. On the other hand, if σ is a face of τ, then a primitive element $n \in N$ is uniquely determined modulo $N \cap \mathbf{R}\sigma$ so that $\tau + (-\sigma) = \mathbf{R}_{\ge 0}n + \mathbf{R}\sigma$. $M \cap \tau^\perp$ is a $\mathbf{Z}$-submodule of corank one in $M \cap \sigma^\perp$. The (σ,τ)-component of d_{II} in this case is then defined to be the tensor product of the restriction homomorphism $\mathcal{O}_{V(\sigma)} \to \mathcal{O}_{V(\tau)}$ with the interior product with respect to n. Namely, an element in the σ-component of $\mathcal{L}_X^{p,q}$ of the form

$$t^m \otimes m_1 \wedge m_2 \wedge \cdots \wedge m_{p-q}, \qquad m, m_1 \in M \cap \sigma^\perp, \quad m_2, \ldots, m_{p-q} \in M \cap \tau^\perp$$

is sent to $t^m \otimes \langle m_1, n\rangle m_2 \wedge \cdots \wedge m_{p-q}$ if $m \in M \cap \tau^\perp$, and to 0 otherwise. d_{II} is the Poincaré residue map.

$d_{\mathrm{I}} \circ d_{\mathrm{I}} = 0$ and $d_{\mathrm{I}} \circ d_{\mathrm{II}} + d_{\mathrm{II}} \circ d_{\mathrm{I}} = 0$ are obvious, while $d_{\mathrm{II}} \circ d_{\mathrm{II}} = 0$ was shown in Ishida [15, Lemma 1.4 and Prop. 1.6]. Consequently, we get a double complex $\mathcal{L}_X^{\cdot,\cdot}$ of $\mathcal{O}_X$-modules, which we call the *logarithmic double complex* for the toric variety X. The associated single complex is denoted by $\mathcal{L}_X^{\cdot}$ and is called the *logarithmic complex* for X.

For simplicity, we denote by $\mathcal{X} := X^{\mathrm{an}}$ the complex analytic space associated to the toric variety $X = T_N\,\mathrm{emb}(\Delta)$.

Theorem 1.1 *If Δ is a* simplicial *fan, then we have a quasi-isomorphism* $\mathbf{C}_{\mathcal{X}} \simeq (\mathcal{L}_X^{\cdot\cdot})^{\mathrm{an}}$.

Remark. In [19], $\mathcal{L}_X^{p,\cdot}$ was denoted by $\mathcal{K}^{\cdot}(X;p)$.

As for an *arbitrary* fan Δ for $N \cong (\mathbf{Z})^r$ which need not be simplicial, Ishida has a proof for the following amazing result: $(\mathcal{L}_X^{\cdot\cdot})^{\mathrm{an}}[2r]$ is quasi-isomorphic to the *globally normalized dualizing complex* $\mathcal{D}_{\mathcal{X}}^{\cdot}$ of $\mathbf{C}_{\mathcal{X}}$-modules in the sense of Verdier [29]. The point of this result of Ishida's lies in the fact that the dualizing complex $\mathcal{D}_{\mathcal{X}}^{\cdot}$ can be expressed in terms of a complex comprizing of *algebraic* and *coherent* $\mathcal{O}_X$-modules. Analogously, Ishida [15, Theorem 3.3] and [16, Theorem 5.4] earlier showed $\mathcal{L}_X^{r,\cdot}[r]$ to be quasi-isomorphic to the *globally normalized dualizing complex* of $\mathcal{O}_X$-modules.

1.2 Ishida's complexes

In this subsection, we recall the definition of Ishida's complexes by quoting the relevant part of [22].

Let Δ be a finite fan for a free $\mathbf{Z}$-module N of rank r, and for $0 \leq q \leq r$ denote $\Delta(q) := \{\sigma \in \Delta \mid \dim \sigma = q\}$ as before.

For each integer p with $0 \leq p \leq r$, *Ishida's p-th complex* $C^{\cdot}(\Delta, \Lambda^p)$ of $\mathbf{Z}$-modules is defined as follows:

$$C^q(\Delta, \Lambda^p) := \bigoplus_{\sigma\in\Delta(q)} \bigwedge^{p-q}(M \cap \sigma^{\perp}) \qquad \text{if} \quad 0 \leq q \leq p,$$

and $C^q(\Delta, \Lambda^p) = 0$ otherwise. For $\sigma \in \Delta(q)$ and $\tau \in \Delta(q+1)$, the (σ, τ)-component of the coboundary map

$$\delta : C^q(\Delta, \Lambda^p) = \bigoplus_{\sigma\in\Delta(q)} \bigwedge^{p-q}(M \cap \sigma^{\perp}) \longrightarrow C^{q+1}(\Delta, \Lambda^p) = \bigoplus_{\tau\in\Delta(q+1)} \bigwedge^{p-q-1}(M \cap \tau^{\perp})$$

is defined to be 0 if σ is not a face of τ. On the other hand, if σ is a face of τ, then a primitive element $n \in N$ is uniquely determined modulo $N \cap \mathbf{R}\sigma$ so that $\tau + (-\sigma) = \mathbf{R}_{\geq 0}n + \mathbf{R}\sigma$. The (σ, τ)-component of δ in this case is defined to be the interior product with respect to this n. Namely, the element $m_1 \wedge m_2 \wedge \cdots \wedge m_{p-q}$ with $m_1 \in M \cap \sigma^{\perp}$ and $m_2, \ldots, m_{p-q} \in M \cap \tau^{\perp}$ is sent to $\langle m_1, n\rangle m_2 \wedge \cdots \wedge m_{p-q}$. As Ishida [15, Prop. 1.6] showed, $\delta \circ \delta = 0$ holds so that $C^{\cdot}(\Delta, \Lambda^p)$ is a complex of $\mathbf{Z}$-modules. We denote its cohomology group by $H^{\cdot}(\Delta, \Lambda^p)$. We will be mainly concerned with their scalar extensions $C^{\cdot}(\Delta, \Lambda^p)_{\mathbf{Q}}$, $C^{\cdot}(\Delta, \Lambda^p)_{\mathbf{C}}$, $H^{\cdot}(\Delta, \Lambda^p)_{\mathbf{Q}}$, $H^{\cdot}(\Delta, \Lambda^p)_{\mathbf{C}}$ to $\mathbf{Q}$ and $\mathbf{C}$.

By definition, we have $H^q(\Delta, \Lambda^p) = 0$ unless $0 \leq q \leq p$.

Remark. In [19, §3.2], $C^{\cdot}(\Delta, \Lambda^p)$ and $H^{\cdot}(\Delta, \Lambda^p)$ were denoted by $C^{\cdot}(\Delta; p)$ and $H^{\cdot}(\Delta; p)$, respectively.

As in [20], we can define a similar complex $C^{\cdot}(\Pi, \mathcal{G}_p)$ of $\mathbf{R}$-vector spaces for a simplicial polyhedral cone decomposition Π of an $\mathbf{R}$-vector space endowed with a marking for each one-dimensional cone in Π. Note, however, that unless a lattice N is given as in the case of a fan, we cannot define the coboundary map in the case of a *non-simplicial* convex polyhedral cone decomposition Π even if it is endowed with a marking. It is crucial that for a codimension one face σ of τ in a fan, a primitive element $n \in N$ is uniquely determined modulo $N \cap \mathbf{R}\sigma$ so that $\tau + (-\sigma) = \mathbf{R}_{\geq 0} n + \mathbf{R}\sigma$ holds as above, regardless of whether τ is simplicial or not.

The following result is slightly stronger than [19, Lemma 3.7], and the proof is similar to that for [20, Prop. 3.5], which concerns analogous $\mathbf{R}$-coefficient cohomology groups for the simplicial polyhedral cone decomposition consisting of the faces of a simplicial cone in a finite dimensional $\mathbf{R}$-vector space:

Proposition 1.2 *Let π be a* simplicial *rational polyhedral cone in $N_{\mathbf{R}}$. Then for each $0 \leq p \leq r$, the cohomology group of Ishida's p-th complex for the fan Γ_π consisting of all the faces of π satisfies*

$$H^q(\Gamma_\pi, \Lambda^p)_{\mathbf{Q}} := H^q(\Gamma_\pi, \Lambda^p) \otimes_{\mathbf{Z}} \mathbf{Q} = \begin{cases} \bigwedge^p (M_{\mathbf{Q}} \cap \pi^{\perp}) & q = 0 \\ 0 & q \neq 0, \end{cases}$$

where $M_{\mathbf{Q}} := M \otimes_{\mathbf{Z}} \mathbf{Q}$.

1.3 The algebraic de Rham theorem

In this subsection, we denote by $X := T_N \operatorname{emb}(\Delta)$ the r-dimensional toric variety over $\mathbf{C}$ corresponding to a finite fan Δ for $N \cong \mathbf{Z}^r$. For simplicity, we again denote the corresponding complex analytic space by $\mathcal{X} := X^{\mathrm{an}}$.

Proposition 1.3 *For an* arbitrary *fan Δ which need not be complete nor simplicial, the hypercohomology group of the logarithmic complex $\mathcal{L}_X^{\cdot}$ has a direct sum decomposition*

$$\mathbf{H}^l(X, \mathcal{L}_X^{\cdot}) = \bigoplus_{p+q=l} H^q(\Delta, \Lambda^p)_{\mathbf{C}} \qquad \text{for each } l.$$

We are now ready to state a generalization, in the toric context, of the algebraic de Rham theorem due to Grothendieck [13]. An algebro-geometric proof valid in the case of complete nonsingular fans can be found in Danilov [4] and in [19, Theorem 3.11].

Theorem 1.4 (The algebraic de Rham theorem) *For a* simplicial *fan* Δ *which need not be complete, we have the following for each* l*:*

$$H^l(\mathcal{X}, \mathbf{C}) = \mathbf{H}^l(\mathcal{X}, (\mathcal{L}^{\cdot}_X)^{\mathrm{an}}) = \mathbf{H}^l(X, \mathcal{L}^{\cdot}_X) = \bigoplus_{p+q=l} H^q(\Delta, \Lambda^p)_{\mathbf{C}},$$

where the second term from the left is the corresponding analytic hypercohomology group.

1.4 Vanishing theorems

Analogs of the following were proved first in [19, Theorem 3.11] for complete nonsingular fans by means of the algebraic de Rham theorem, and then directly in [20, Theorem 4.1] for complete simplicial polyhedral cone decompositions endowed with markings:

Proposition 1.5 *Let* Δ *be a* simplicial *and* complete *fan for* $N \cong \mathbf{Z}^r$*. Then for all* p *with* $0 \leq p \leq r$ *we have*

$$H^q(\Delta, \Lambda^p)_{\mathbf{Q}} = 0 \qquad \text{for} \quad q \neq p.$$

Moreover for all p*, we have a perfect pairing*

$$H^p(\Delta, \Lambda^p)_{\mathbf{Q}} \times H^{r-p}(\Delta, \Lambda^{r-p})_{\mathbf{Q}} \longrightarrow H^r(\Delta, \Lambda^r)_{\mathbf{Q}} \cong \bigwedge^r M_{\mathbf{Q}}.$$

The following is an important generalization, due to Ishida, of our earlier result stated as Corollary 1.7 later.

Theorem 1.6 (Ishida) *Let* Δ *be a finite simplicial fan for* $N \cong \mathbf{Z}^r$ *which may not be complete. If there exist a finite complete simplicial fan* $\tilde{\Delta}$ *and a* $\rho \in \tilde{\Delta}(1)$ *such that* $\Delta = \tilde{\Delta} \setminus \mathrm{Star}_\rho(\tilde{\Delta})$*, then*

$$H^q(\Delta, \Lambda^p)_{\mathbf{Q}} = 0 \qquad \text{for all} \quad q \neq p.$$

Corollary 1.7 *Let* Δ *be a finite simplicial fan for* $N \cong \mathbf{Z}^r$ *such that its support* $|\Delta|$ *is convex of dimension* r*. Then*

$$H^q(\Delta, \Lambda^p)_{\mathbf{Q}} = 0 \qquad \text{for all} \quad q \neq p.$$

1.5 Chow rings for simplicial fans

It is convenient to introduce the Chow ring over $\mathbf{Q}$ for simplicial fans, generalizing results in Danilov [4] and [19], [20].

Let Δ be a finite *simplicial* fan for $N \cong \mathbf{Z}^r$ which need not be complete. As before, we denote $\Delta(1) := \{\rho \in \Delta \mid \dim \rho = 1\}$. For each $\rho \in \Delta(1)$, let $n(\rho)$ be the unique primitive element of N contained in ρ.

Introduce a variable x_ρ for each $\rho \in \Delta(1)$ and denote by

$$S := \mathbf{Q}\,[x_\rho \mid \rho \in \Delta(1)]$$

the polynomial ring over $\mathbf{Q}$ in the variables $\{x_\rho \mid \rho \in \Delta(1)\}$. Let I be the ideal of S generated by the set

$$\{x_{\rho_1}x_{\rho_2}\cdots x_{\rho_s} \mid \rho_1,\ldots,\rho_s \in \Delta(1) \text{ distinct and } \rho_1+\cdots+\rho_s \notin \Delta\}$$

of square-free monomials. On the other hand, let J be the ideal of S generated by the set

$$\left\{ \sum_{\rho\in\Delta(1)} \langle m, n(\rho)\rangle x_\rho \;\middle|\; m \in M \right\}$$

of linear forms.

Definition. The *Chow ring* for a finite simplicial fan Δ is defined to be

$$A = A(\Delta) := S/(I+J).$$

We denote by $v(\rho)$ the image in A of the variable x_ρ. More generally, for each $\sigma \in \Delta$, which is uniquely expressed in the form $\sigma = \rho_1 + \cdots + \rho_p$ with distinct $\rho_1, \ldots, \rho_p \in \Delta(1)$ and $p := \dim\sigma$, we denote $v(\sigma) := v(\rho_1)v(\rho_2)\cdots v(\rho_p)$, which is the image in A of $x_{\rho_1}x_{\rho_2}\cdots x_{\rho_p}$.

Proposition 1.8 *The Chow ring $A = A(\Delta)$ for a simplicial fan Δ for $N \cong \mathbf{Z}^r$ is an Artinian graded algebra of the form*

$$A = \bigoplus_{p=0}^{r} A^p \qquad \textit{with} \quad A^p = A^p(\Delta) = \sum_{\sigma\in\Delta(p)} \mathbf{Q}v(\sigma)$$

and is generated by A^1 over $A^0 = \mathbf{Q}$. Moreover, we have the following relations:

$$\sum_{\rho\in\Delta(1)} \langle m, n(\rho)\rangle v(\rho) = 0 \qquad \textit{for all} \quad m \in M,$$

and, for $\sigma, \sigma' \in \Delta$,

$$v(\sigma)v(\sigma') = \begin{cases} 0 & \textit{if } \sigma+\sigma' \notin \Delta \\ v(\sigma+\sigma') & \textit{if } \sigma\cap\sigma' = \{0\}, \quad \sigma+\sigma' \in \Delta. \end{cases}$$

The following is easy to show:

Lemma 1.9 *For a finite simplicial fan Δ, we have canonical isomorphisms*

$$H^p(\Delta, \Lambda^p)_{\mathbf{Q}} = A^p(\Delta) \qquad \textit{for all} \quad 0 \le p \le r.$$

In view of Theorem 1.4 and Proposition 1.5, we thus have:

Corollary 1.10 *The Chow ring $A(\Delta)$ for a finite simplicial and* complete *fan Δ is a Gorenstein ring and satisfies the duality $\dim_{\mathbf{Q}} A^p(\Delta) = \dim_{\mathbf{Q}} A^{r-p}(\Delta)$ for all $0 \leq p \leq r$. Moreover, $H^l(\mathcal{X}, \mathbf{C}) = 0$ for all l odd, while*

$$H^{2p}(\mathcal{X}, \mathbf{C}) = \mathbf{H}^{2p}(X, \mathcal{L}_X^{\cdot}) = A^p(\Delta)_{\mathbf{C}} \qquad \textit{for all} \quad 0 \leq p \leq r.$$

Corollary 1.11 *If Δ is a finite simplicial fan for $N \cong \mathbf{Z}^r$ such that the support $|\Delta|$ is a convex cone of dimension r, then $H^l(\mathcal{X}, \mathbf{C}) = 0$ for all l odd, while*

$$H^{2p}(\mathcal{X}, \mathbf{C}) = \mathbf{H}^{2p}(X, \mathcal{L}_X^{\cdot}) = A^p(\Delta)_{\mathbf{C}} \qquad \textit{for all} \quad 0 \leq p \leq r.$$

For a simplicial and complete fan Σ for $\bar{N} \cong \mathbf{Z}^{r-1}$, let us now consider equivariant $\mathbf{P}_1(\mathbf{C})$-bundles over $\bar{X} := T_{\bar{N}} \operatorname{emb}(\Sigma)$ and associated $\mathbf{C}$-bundles and $\mathbf{C}^*$-bundles.

For that purpose, let $\eta : \bar{N}_{\mathbf{R}} \to \mathbf{R}$ be an $\mathbf{R}$-valued function which is $\mathbf{Z}$-valued on $\bar{N}$ and piecewise linear with respect to the fan Σ. Denote $N := \bar{N} \oplus \mathbf{Z}n_0 \cong \mathbf{Z}^r$ and consider the graph $g : \bar{N}_{\mathbf{R}} \to N_{\mathbf{R}}$ of η defined by $g(\bar{n}) := \bar{n} + \eta(\bar{n})n_0$. We then let

$$\begin{aligned}
\Phi^b &:= \{g(\bar{\sigma}) \mid \bar{\sigma} \in \Sigma\} \\
\Phi &:= \Phi^b \coprod \left\{\tau + \mathbf{R}_{\geq 0} n_0 \mid \tau \in \Phi^b\right\} \\
\tilde{\Phi} &:= \Phi \coprod \left\{\tau + \mathbf{R}_{\geq 0}(-n_0) \mid \tau \in \Phi^b\right\}.
\end{aligned}$$

The projection $N \to \bar{N}$ killing n_0 induces maps of fans $(N, \tilde{\Phi}) \to (\bar{N}, \Sigma)$, $(N, \Phi) \to (\bar{N}, \Sigma)$ and $(N, \Phi^b) \to (\bar{N}, \Sigma)$ which respectively give an equivariant $\mathbf{P}_1(\mathbf{C})$-bundle $T_N \operatorname{emb}(\tilde{\Phi}) \to \bar{X}$, the associated $\mathbf{C}$-bundle $T_N \operatorname{emb}(\Phi) \to \bar{X}$ and the associated $\mathbf{C}^*$-bundle $T_N \operatorname{emb}(\Phi^b) \to \bar{X}$.

Proposition 1.12 *$A(\tilde{\Phi})$ is canonically isomorphic to the algebra $A(\Sigma)[\xi]$ over $A(\Sigma)$ generated by an element ξ subject to the relation*

$$\xi(\xi + \bar{\eta}) = 0 \qquad \textit{with} \quad \bar{\eta} := \sum_{\bar{\rho} \in \Sigma(1)} \eta(\bar{n}(\bar{\rho}))\bar{v}(\bar{\rho}) \in A^1(\Sigma),$$

where $\bar{n}(\bar{\rho})$ and $\bar{v}(\bar{\rho})$ for $\bar{\rho} \in \Sigma(1)$ are similar to $n(\rho)$ and $v(\rho)$ previously defined for $\rho \in \Delta(1)$.

Moreover, we have canonical isomorphisms

$$A(\Phi) = A(\Sigma) \qquad \textit{and} \qquad A(\Phi^b) = A(\Sigma)/A(\Sigma)\bar{\eta}.$$

In fact, we have further information which we need later:

Proposition 1.13 *We have $H^q(\Phi^b, \Lambda^p)_{\mathbf{Q}} = 0$ for all $q \neq p-1, p$. Furthermore, for all p, there exists a canonical exact sequence*

$$0 \to H^{p-1}(\Phi^b, \Lambda^p)_{\mathbf{Q}} \to A^{p-1}(\Sigma) \xrightarrow{\bar{\eta}} A^p(\Sigma) \to H^p(\Phi^b, \Lambda^p)_{\mathbf{Q}} = A^p(\Phi^b) \to 0,$$

where the arrow with $\bar{\eta}$ denotes the multiplication by $\bar{\eta} \in A^1(\Sigma)$.

2 The intersection cohomology

2.1 Intersection cohomology and intersection complex

We briefly recall definitions and relevant results on the intersection cohomology and intersection complex. We restrict ourselves to those over $\mathbf{C}$ and with respect to the middle perversity. For details, we refer the reader to Beilinson, Bernstein and Deligne [1], Borel et al. [2], Brylinski [3], Deligne [8], Goresky and MacPherson [10], [11], [12], Kirwan [17] and others.

Let Y be an r-dimensional normal algebraic variety (of finite type) over the field $\mathbf{C}$ of complex numbers, and denote by $\mathcal{Y} := Y^{\mathrm{an}}$ the associated complex analytic space.

The intersection complex of $\mathbf{C}_{\mathcal{Y}}$-modules (with respect to the middle perversity) is an object $\mathcal{IC}^{\cdot}_{\mathcal{Y}}$ in the derived category $\mathrm{D}^b_c(\mathbf{C}_{\mathcal{Y}})$ of bounded complexes of $\mathbf{C}_{\mathcal{Y}}$-modules with algebraically constructible cohomology sheaves. We here adopt the convention on the degrees so that the homology sheaf satisfies

$$\mathcal{H}^l(\mathcal{IC}^{\cdot}_{\mathcal{Y}}) = 0 \qquad \text{unless} \quad -2r \leq l \leq 0.$$

The intersection cohomology group with coefficients in $\mathbf{C}$ and with respect to the middle perversity is defined to be the hypercohomology group

$$IH^{\cdot}(\mathcal{Y}, \mathbf{C}) := \mathbf{H}^{\cdot}(\mathcal{Y}, \mathcal{IC}^{\cdot}_{\mathcal{Y}}[-2r]),$$

non-vanishing terms of which thus occur only in degrees between 0 and $2r$. The intersection cohomology group with compact support is defined similarly by

$$IH^{\cdot}_c(\mathcal{Y}, \mathbf{C}) := \mathbf{H}^{\cdot}_c(\mathcal{Y}, \mathcal{IC}^{\cdot}_{\mathcal{Y}}[-2r]).$$

Remark. The intersection complex of $\mathbf{C}_{\mathcal{Y}}$-modules with respect to the zero perversity is $\mathbf{C}_{\mathcal{Y}}[2r]$, while that with respect to the top perversity is the globally normalized dualizing complex $\mathcal{D}^{\cdot}_{\mathcal{Y}}$ of $\mathbf{C}_{\mathcal{Y}}$-modules in the sense of Verdier (cf. [29]).

Theorem 2.1 (Unique characterization, cf. Goresky and MacPherson [12, §4.1 and §4.3]) *$\mathcal{IC}^{\cdot}_{\mathcal{Y}}$ is a unique object in $\mathbf{D}^b_c(\mathbf{C}_{\mathcal{Y}})$ satisfying the following properties:*

- $\mathcal{IC}^{\cdot}_{\mathcal{Y}}|_{\mathcal{Y}^\circ} = \mathbf{C}_{\mathcal{Y}^\circ}[2r]$ *for a Zariski dense nonsingular open subset* $\mathcal{Y}^\circ \subset \mathcal{Y}$.
- $\mathcal{H}^j(\mathcal{IC}^{\cdot}_{\mathcal{Y}}) = 0$ *for* $j \leq -2r$.
- $\dim_{\mathbf{C}} \operatorname{supp} \mathcal{H}^j(\mathcal{IC}^{\cdot}_{\mathcal{Y}}) < -j - r$ *for* $-2r < j$.
- *(Verdier self-duality)* $\mathcal{D}_{\mathcal{Y}}(\mathcal{IC}^{\cdot}_{\mathcal{Y}}) = \mathcal{IC}^{\cdot}_{\mathcal{Y}}[-2r]$.

Theorem 2.2 (Special case of the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber, cf. Beilinson, Bernstein and Deligne [1]; Goresky and MacPherson [11]) *Let $f : Y' \to Y$ be a* proper birational *morphism of normal varieties. Then $\mathcal{IC}^{\cdot}_{\mathcal{Y}}$ is a direct factor of the direct image $\mathbf{R}f^{\mathrm{an}}_*\mathcal{IC}^{\cdot}_{\mathcal{Y}'}$ as objects in $\mathbf{D}^b_c(\mathbf{C}_{\mathcal{Y}})$. In particular, the equality $\mathcal{IC}^{\cdot}_{\mathcal{Y}} = \mathbf{R}f^{\mathrm{an}}_*\mathcal{IC}^{\cdot}_{\mathcal{Y}'}$ holds if Y' is nonsingular (or, more generally, with at worst quotient singularities) and f is* small, *i.e.,*

$$\operatorname{codim}_{\mathbf{C}}\{y \in Y \mid \dim_{\mathbf{C}} f^{-1}(y) \geq l\} > 2l$$

holds for all $l > 0$.

Let $X := T_N \operatorname{emb}(\Delta)$ be the r-dimensional toric variety over $\mathbf{C}$ corresponding to a finite fan Δ for $N \cong \mathbf{Z}^r$, and denote $\mathcal{X} := X^{\mathrm{an}}$.

Combining a result due to Steenbrink [28] with our Theorem 1.1 and the algebraic de Rham theorem (Theorem 1.4), we have:

Theorem 2.3 *If Δ is* simplicial, *then X has at worst quotient singularities so that*

$$\mathbf{C}_{\mathcal{X}}[2r] = \mathcal{IC}^{\cdot}_{\mathcal{X}} = \mathcal{D}^{\cdot}_{\mathcal{X}} = (\mathcal{L}^{\cdot}_X)^{\mathrm{an}}[2r]$$

holds in $\mathbf{D}^b_c(\mathbf{C}_{\mathcal{X}})$. Hence for all l we have

$$IH^l(\mathcal{X}, \mathbf{C}) = H^l(\mathcal{X}, \mathbf{C}) = \mathbf{H}^l(\mathcal{X}, (\mathcal{L}^{\cdot}_X)^{\mathrm{an}}) = \mathbf{H}^l(X, \mathcal{L}^{\cdot}_X) = \bigoplus_{p+q=l} H^q(\Delta, \Lambda^p)_{\mathbf{C}}.$$

In view of Corollary 1.7, Corollary 1.11, and the decomposition theorem (Theorem 2.2), we have:

Corollary 2.4 *Let Δ be a finite fan for $N \cong \mathbf{Z}^r$ which need not be simplicial nor complete. If the support $|\Delta|$ is a convex cone of dimension r, then*

$$IH^l(\mathcal{X}, \mathbf{C}) = 0 \qquad \textit{for } l \textit{ odd.}$$

holds for $X := T_N \operatorname{emb}(\Delta)$. If Δ is simplicial, *then for all $0 \leq p \leq r$ we have*

$$IH^{2p}(\mathcal{X}, \mathbf{C}) = H^{2p}(\mathcal{X}, \mathbf{C}) = \mathbf{H}^{2p}(X, \mathcal{L}^{\cdot}_X) = A^p(\Delta)_{\mathbf{C}}.$$

For the proof, choose a simplicial subdivision Δ' of Δ and apply Theorem 2.3 to Δ'. We are done, since $IH^l(\mathcal{X}, \mathbf{C})$ is a subspace of $IH^l(\mathcal{X}', \mathbf{C})$ by the decomposition theorem.

The following in straightforward:

Lemma 2.5 *Let a finite fan Δ' be a* simplicial *subdivision of a fan Δ for N. Then the corresponding equivariant morphism $f : X' := T_N \operatorname{emb}(\Delta') \to X := T_N \operatorname{emb}(\Delta)$ of toric varieties is small (that is,*

$$\operatorname{codim}_{\mathbf{C}}\{x \in X \mid \dim_{\mathbf{C}} f^{-1}(x) \geq l\} > 2l$$

holds for all $l > 0$) if and only if the following holds for all those σ in Δ which are actually *subdivided in Δ':*

$$\min\{\dim \sigma' \mid \sigma' \in \Delta', \sigma' \subset \sigma, \sigma' \cap \operatorname{rel\,int} \sigma \neq \emptyset\} > \frac{1}{2} \dim \sigma.$$

In this case, Δ' is said to be a small simplicial subdivision *of Δ.*

2.2 Isolated toric singularities

Let us look at the intersection cohomology and the intersection complex in the simplest nontrivial case.

Let π be a strongly convex rational polyhedral cone in $N_{\mathbf{R}} \cong \mathbf{R}^r$ with $\dim \pi = r$ such that π itself is not simplicial but all the proper faces of π are simplicial. Let us denote

$$\Gamma_\pi := \{\text{the faces of } \pi\}, \qquad \partial\pi := \{\text{all the proper faces of } \pi\}.$$

Then

$$U_\pi := T_N \operatorname{emb}(\Gamma_\pi) = \operatorname{Spec}(\mathbf{C}[M \cap \pi^\vee])$$

is a toric variety with a unique isolated non-quotient singular point $P := \operatorname{orb}(\pi)$. By assumption, $\partial\pi$ is simplicial so that $U_\pi \setminus \{P\} = T_N \operatorname{emb}(\partial\pi)$ has at worst quotient singularities.

Let us denote $\mathcal{U} := (U_\pi)^{\mathrm{an}}$ and study $\mathcal{IC}^\cdot_{\mathcal{U}}$, $IH^\cdot(\mathcal{U}, \mathbf{C})$ and $IH^\cdot_c(\mathcal{U}, \mathbf{C})$.

Lemma 2.6 (cf. Denef and Loeser [9, Cor. 6.6 and Lemma 6.7]) *In the notation above, the stalk of $\mathcal{H}^\cdot(\mathcal{IC}^\cdot_{\mathcal{U}}[-2r])$ at P satisfies $\mathcal{H}^\cdot(\mathcal{IC}^\cdot_{\mathcal{U}}[-2r])_P = IH^\cdot(\mathcal{U}, \mathbf{C})$. In particular, we have $IH^l(\mathcal{U}, \mathbf{C}) = 0$ for $l \geq r$ in view of the unique characterization in Theorem* 2.1. *Moreover, the inclusion $\mathcal{U} \setminus \{P\} \to \mathcal{U}$ induces isomorphisms*

$$IH^l(\mathcal{U}, \mathbf{C}) \xrightarrow{\sim} H^l(\mathcal{U} \setminus \{P\}, \mathbf{C}) \qquad \textit{for all} \quad l < r.$$

Let us now consider a simplicial subdivision Δ' of π which subdivides none of the proper faces of π. Namely, Δ' is a simplicial fan for N such that $|\Delta'| = \pi$ and $\Delta' \supset \partial\pi$. We denote

$$(\Delta')^\circ := \Delta' \setminus \partial\pi = \{\sigma' \in \Delta' \mid \sigma' \cap \operatorname{int}\pi \neq \emptyset\}.$$

By definition, Δ' is a small simplicial subdivision of π if and only if $\dim\sigma' > r/2$ holds for all $\sigma' \in (\Delta')^\circ$.

Let $f : X' := T_N \operatorname{emb}(\Delta') \to U_\pi$ be the corresponding equivariant proper birational morphism, and denote by $f^{\mathrm{an}} : \mathcal{X}' := (X')^{\mathrm{an}} \to \mathcal{U} := (U_\pi)^{\mathrm{an}}$ the corresponding morphism of analytic spaces.

By Theorem 2.3, we have

$$\mathcal{IC}^\cdot_{\mathcal{X}'}[-2r] = \mathbf{C}_{\mathcal{X}'} = (\mathcal{L}^\cdot_{X'})^{\mathrm{an}}.$$

Hence by Theorem 2.2 (the decomposition theorem), $\mathcal{IC}^\cdot_{\mathcal{U}}[-2r]$ is a direct factor of

$$\mathbf{R}f^{\mathrm{an}}_*(\mathbf{C}_{\mathcal{X}'}) = (\mathbf{R}f_*\mathcal{L}^\cdot_{X'})^{\mathrm{an}}$$

as objects in $\mathbf{D}^b_c(\mathbf{C}_{\mathcal{U}})$. Moreover, if Δ' is a small simplicial subdivision, we have the equality

$$\mathcal{IC}^\cdot_{\mathcal{U}}[-2r] = \mathbf{R}f^{\mathrm{an}}_*(\mathbf{C}_{\mathcal{X}'}) = (\mathbf{R}f_*\mathcal{L}^\cdot_{X'})^{\mathrm{an}}.$$

$(\Delta')^\circ$ is star-closed in Δ' and $\partial\pi = \Delta' \setminus (\Delta')^\circ$. Hence for all p we have a long exact sequence of Ishida's complexes

$$0 \to C^\cdot((\Delta')^\circ, \Lambda^p) \to C^\cdot(\Delta', \Lambda^p) \to C^\cdot(\partial\pi, \Lambda^p) \to 0.$$

In view of the vanishing of Ishida's cohomology in Corollary 1.7 and Proposition 1.13, the non-vanishing part of the associated long exact sequence is

$$0 \to H^{p-1}(\partial\pi, \Lambda^p)_{\mathbf{Q}} \to H^p((\Delta')^\circ, \Lambda^p)_{\mathbf{Q}} \to H^p(\Delta', \Lambda^p)_{\mathbf{Q}} \to H^p(\partial\pi, \Lambda^p)_{\mathbf{Q}} \to 0.$$

By the decomposition theorem, $IH^{2p}(\mathcal{U}, \mathbf{C})$ is a subspace of

$$\mathbf{H}^{2p}(\mathcal{U}, \mathbf{R}f^{\mathrm{an}}_*(\mathbf{C}_{\mathcal{X}'})) = H^{2p}(\mathcal{X}', \mathbf{C}) = H^p(\Delta', \Lambda^p)_{\mathbf{C}} = A^p(\Delta')_{\mathbf{C}}.$$

On the other hand, $U_\pi \setminus \{P\} = T_N \operatorname{emb}(\partial\pi)$ with $\partial\pi$ simplicial. Hence by Theorem 1.4 and Proposition 1.13, we have

$$\begin{aligned} IH^{2p}(\mathcal{U} \setminus \{P\}, \mathbf{C}) &= H^{2p}(\mathcal{U} \setminus \{P\}, \mathbf{C}) = H^p(\mathcal{U} \setminus \{P\}, \Lambda^p)_{\mathbf{C}} = A^p(\partial\pi)_{\mathbf{C}} \\ IH^{2p-1}(\mathcal{U} \setminus \{P\}, \mathbf{C}) &= H^{2p-1}(\mathcal{U} \setminus \{P\}, \mathbf{C}) = H^{p-1}(\partial\pi, \Lambda^p)_{\mathbf{C}}. \end{aligned}$$

Proposition 2.7 *We have $IH^l(\mathcal{U}, \mathrm{C}) = 0$ for l odd or $l \geq r$. For $l = 2p$, $IH^{2p}(\mathcal{U}, \mathrm{C})$ is a subspace of $A^p(\Delta')_{\mathrm{C}}$ which splits the canonical surjection*

$$A^p(\Delta')_{\mathrm{C}} \longrightarrow A^p(\partial\pi)_{\mathrm{C}} \to 0.$$

If Δ' is small, then the above surjection is an isomorphism, and $IH^{2p}(\mathcal{U}, \mathrm{C}) = A^p(\Delta')_{\mathrm{C}} = A^p(\partial\pi)_{\mathrm{C}}$.

In view of Lemma 2.6 due to Denef and Loeser, this proposition follows from the vanishing

$$A^p(\partial\pi)_{\mathrm{C}} = 0 \qquad \text{for} \quad p > (r-1)/2,$$

which is equivalent to the strong Lefschetz theorem for projective toric varieties with at worst quotient singularities, as we now see.

Choose a primitive element $n_0 \in N$ which is contained in the interior of π. Then there certainly exists a decomposition $N = \bar{N} \oplus \mathbf{Z}n_0$ with $\bar{N} \cong \mathbf{Z}^{r-1}$. Since π is assumed to be a strongly convex cone, there exist a complete fan Σ for $\bar{N}$ and an $\mathbf{R}$-valued function $\eta : \bar{N}_{\mathbf{R}} \to \mathbf{R}$ which is $\mathbf{Z}$-valued on $\bar{N}$ and is piecewise linear and strictly convex with respect to Σ such that

$$\partial\pi = \{g(\bar{\sigma}) \mid \bar{\sigma} \in \Sigma\} = \Phi^b$$

as in Subsection 1.5, where $g : \bar{N}_{\mathbf{R}} \to N_{\mathbf{R}}$ is the graph of η defined by $g(\bar{n}) := \bar{n} + \eta(\bar{n})n_0$. Hence by Proposition 1.13 we have

$$A^p(\partial\pi) = A^p(\Sigma)/\bar{\eta}A^{p-1}(\Sigma) \qquad \text{with} \quad \bar{\eta} := \sum_{\bar{\rho} \in \Sigma(1)} \eta(\bar{n}(\bar{\rho}))\bar{v}(\bar{\rho}) \in A^1(\Sigma).$$

Its vanishing in degrees $p > (r-1)/2$ is exactly the strong Lefschetz theorem for the projective toric variety Σ with respect to the ample element $\bar{\eta} \in A^1(\Sigma)$ by [20, Cor. 4.5].

In the same notation, let

$$\Phi := \Phi^b \coprod \{\tau + \mathbf{R}_{\geq 0}n_0\},$$

which is a simplicial subdivision of π not subdividing any of the proper faces of π. Thus we may take this Φ as Δ' in Proposition 2.7. Since $A(\Phi) = A(\Sigma)$ by Proposition 1.12, We have:

Corollary 2.8 *For each p, the intersection cohomology group $IH^{2p}(\mathcal{U}, \mathrm{C})$ is a subspace of $A^p(\Sigma)_{\mathrm{C}}$ which gives a splitting of the surjection*

$$A^p(\Sigma) \to A^p(\Sigma)/\bar{\eta}A^{p-1}(\Sigma).$$

As to the existence and various possibilities for simplicial subdivisions Δ' of π subdividing none of the proper faces of π, we have the following results:

(1) Φ constructed above is an example. In this case, the set of one-dimensional faces is $\Phi(1) = (\partial\pi)(1) \coprod \{\mathbf{R}_{\geq 0} n_0\}$ (see Figure 1).

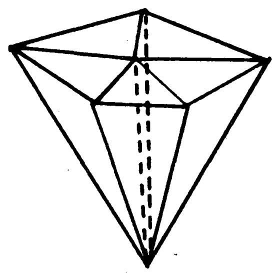

Figure 1: Φ

(2) As an application of the Gelfand-Kapranov-Zelevinskij decompositions, we have the following (see [23, Cor. 3.8] and Park's account in these proceedings): There exist *non-divisorial* Δ', namely, those which do not introduce any new one-dimensional cones other than the one-dimensional faces of π so that $\Delta'(1) = (\partial\pi)(1)$, hence $\dim \sigma > 1$ for any $\sigma \in (\Delta')^{\circ}$. Moreover, any two such non-divisorial simplicial subdivisions can be obtained from each other by a finite succession of *flops*.

(3) Some of the non-divisorial Δ' in (2) are small so that $\dim \sigma > r/2$ for any $\sigma \in (\Delta')^{\circ}$, while the others may not be (see Park's account in these proceedings).

For $r = 3$, all the non-divisorial Δ' are small (see Figure 2).

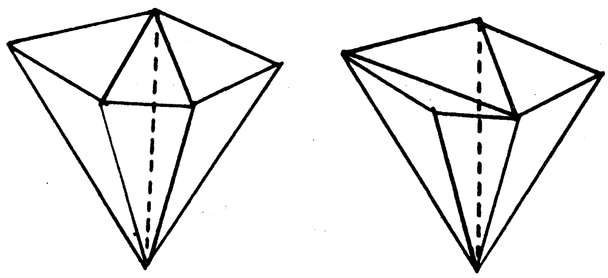

Figure 2: $r = 3$

In the case $r = 4$, suppose the section $\pi \cap H$ of π by a hyperplane H not passing through the origin is a simplicial hexahedron as in Figure 3. Then one of the two non-divisorial simplicial subdivisions is small, while the other is not.

(4) For some π, small simplicial subdivisions may not exist. As an example, let π be a 4-dimensional cone such that the section $\pi \cap H$ of π by a hyperplane H not passing through

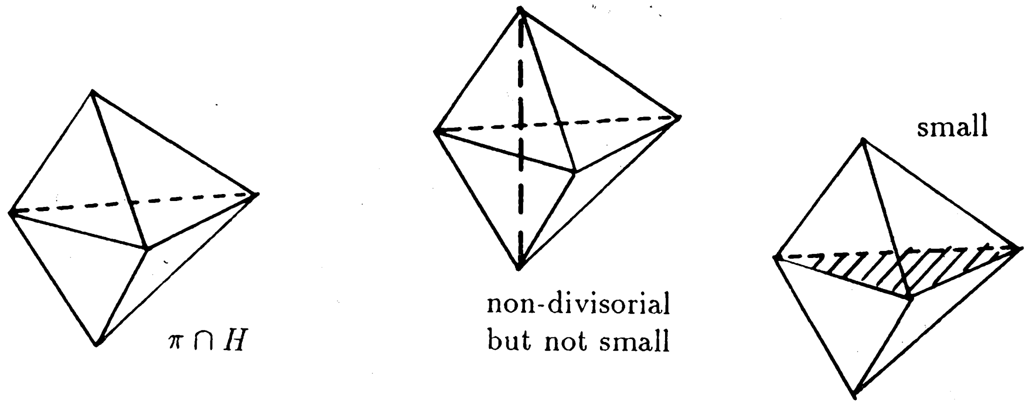

Figure 3: $r = 4$, hexahedral

the origin is a simplicial octahedron as in Figure 4. This π has three different non-divisorial simplicial subdivisions Δ_1', Δ_2', Δ_3' which are not small. The coarsest common subdivision is Φ for an appropriate choice of n_0.

Let us calculate the Chow rings of Φ, Δ_1', Δ_2', Δ_3' for π appearing in (4). For simplicity, we assume that the primitive elements of N corresponding to the central as well as the three pairs of diametrically opposite vertices of $\pi \cap H$ are n_0; n_1, n_1'; n_2, n_2'; n_3, n_3' satisfying

$$n_0 = n_1 + n_1' = n_2 + n_2' = n_3 + n_3',$$

where $\{n_0, n_1, n_2, n_3\}$ is a $\mathbf{Z}$-basis for N. The Chow ring of Φ turns out to be

$$A(\Phi) = \mathbf{Q}[v_1, v_2, v_3] \qquad \text{with} \quad v_1^2 = v_2^2 = v_3^2 = 0,$$

hence

$$\begin{aligned}
A^0(\Phi) &= \mathbf{Q} \\
A^1(\Phi) &= \mathbf{Q}v_1 \oplus \mathbf{Q}v_2 \oplus \mathbf{Q}v_3 \\
A^2(\Phi) &= \mathbf{Q}v_1v_2 \oplus \mathbf{Q}v_2v_3 \oplus \mathbf{Q}v_3v_1 \\
A^3(\Phi) &= \mathbf{Q}v_1v_2v_3 \\
A^4(\Phi) &= \{0\}.
\end{aligned}$$

Moreover, with $v_0 := -(v_1 + v_2 + v_3)$, we have $A(\partial\pi) = A/Av_0$, whose non-vanishing components are

$$A^0(\partial\pi) = \mathbf{Q}, \quad A^1(\partial\pi) = (\mathbf{Q}v_1 \oplus \mathbf{Q}v_2 \oplus \mathbf{Q}v_3)/\mathbf{Q}v_0 \cong \mathbf{Q}^2.$$

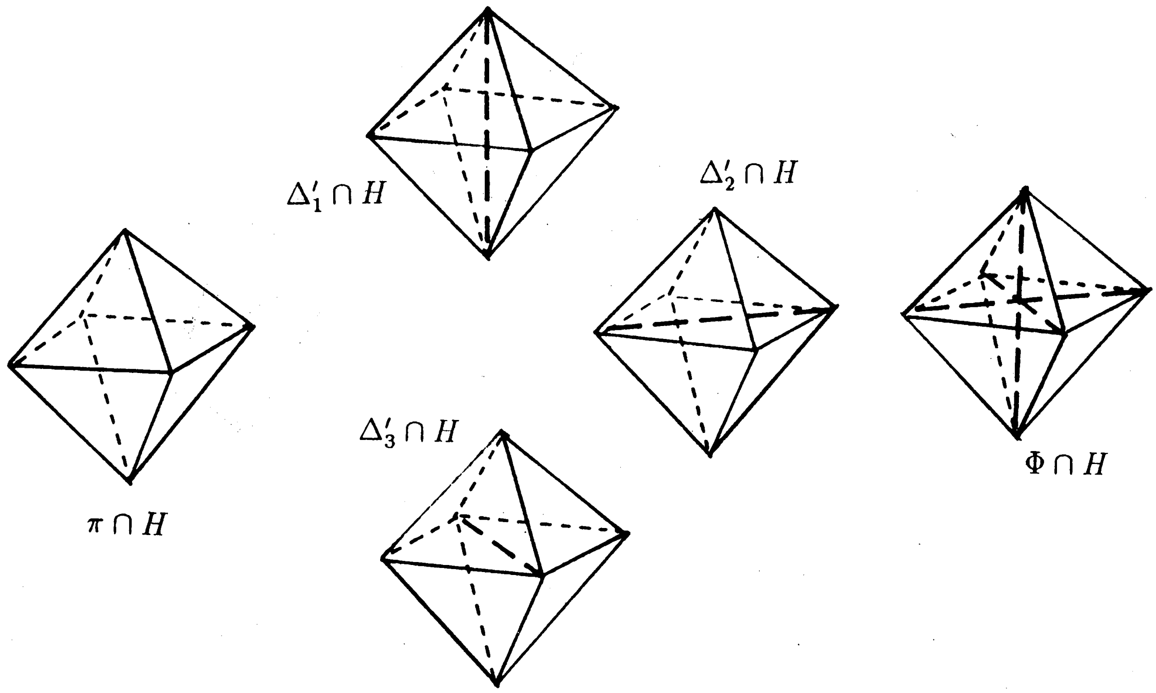

Figure 4: $r = 4$, octahedral

Consequently, we have $IH^2(\mathcal{U}, \mathbf{C}) \cong \mathbf{C}^2$.

Φ is a star subdivision of Δ'_1 with respect to $n_0 \in \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n'_1$, hence we have a natural isomorphism

$$A(\Delta'_1) \xrightarrow{\sim} \mathbf{Q}[v_2, v_3] \subset A(\Phi) = \mathbf{Q}[v_1, v_2, v_3].$$

Similarly, we have

$$\begin{aligned} A(\Delta'_2) &\xrightarrow{\sim} \mathbf{Q}[v_1, v_3] \subset A(\Phi) = \mathbf{Q}[v_1, v_2, v_3] \\ A(\Delta'_3) &\xrightarrow{\sim} \mathbf{Q}[v_1, v_2] \subset A(\Phi) = \mathbf{Q}[v_1, v_2, v_3]. \end{aligned}$$

$IH^2(\mathcal{U}, \mathbf{C})$ as a subspace of $A^1(\Delta'_1)_{\mathbf{C}}$ (resp. $A^1(\Delta'_2)_{\mathbf{C}}$, resp. $A^1(\Delta'_3)_{\mathbf{C}}$) coincides with the whole space, since both are 2-dimensional. However, by the natural isomorphisms above, it is mapped to three different subspaces

$$\mathbf{C}v_2 \oplus \mathbf{C}v_3, \quad \mathbf{C}v_1 \oplus \mathbf{C}v_3, \quad \mathbf{C}v_1 \oplus \mathbf{C}v_2 \subset A^1(\Phi)_{\mathbf{C}} = \mathbf{C}v_1 \oplus \mathbf{C}v_2 \oplus \mathbf{C}v_3.$$

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